Iwasawa λ invaraints and Massey products

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- Iwasawa theory
- Galois cohomology
- $\cdots$

In the talk, we will compare these two methods.

Iwasawa's answer to the question: Instead of looking at one field extension, we look at a tower of field extensions.

Definition

We call a tower of fields extension $K \subset K_1 \subset K_2 \subset \cdots \subset K_l \subset \cdots \subset K_\infty$ a $\mathbb{Z}_p$ extension of number field K if $\mathrm{Gal}(K_l/K) \cong \mathbb{Z}/p^l\mathbb{Z}$ and $\mathrm{Gal}(K_\infty/K) \cong \mathbb{Z}_p$.

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- Generally, by compositing the above tower with number field K, we get a $\mathbb{Z}_p$ extension for K: $K = K\mathbb{Q}_1 = \cdots = K\mathbb{Q}_e \subset K\mathbb{Q}_{e+1} \subset K\mathbb{Q}_{e+2} \cdots \subset K\mathbb{Q}_\infty$

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- We call such $\mathbb{Z}_p$ extension as cyclotomic $\mathbb{Z}_p$ extension.

Iwasawa theory

Let $K \subset K_1 \subset K_2 \subset \cdots \subset K_l \subset \cdots \subset K_\infty$ be a $\mathbb{Z}_p$ extension of number field K.

Theorem (Iwasawa[Was97])

There are constants μ, λ, ν such that when n is sufficient large,

$$\#\mathrm{Cl}(\mathcal{O}_{K_l})[p^\infty] = p^{\mu p^l + \lambda l + \nu}$$

Therefore people are interested in computing the three constants μ, λ, ν

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Conjecture (Iwasawa[Was97])

The Iwasawa μ is zero for the cyclotomic $\mathbb{Z}_p$ extension of any number field K

So, the interesting part is to calculate invariant λ.

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- Let $\alpha \in K^*$, we can also view α as an element in cohomology group $H^1(G_{K,S}, \mu_p)$ by kummer theory

McCallum-Sharifi's result

Recall μ_n is the group of n-th root of units.

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- $\mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$ can act on $\mathrm{Cl}(\mathbb{Q}(\mu_{p^l}))[p^\infty]$.
- Decompose $\mathrm{Cl}(\mathbb{Q}(\mu_{p^l}))[p^\infty] = \oplus_i \varepsilon_i \mathrm{Cl}(\mathbb{Q}(\mu_{p^l}))[p^\infty]$ as direct sum of eigenspaces with respect to the action of $\mathrm{Gal}(\mathbb{Q}(\mu_p))$.

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- By Iwasawa theory

$$\#\varepsilon_i \mathrm{Cl}(\mathbb{Q}(\mu_{p^l}))[p^\infty] = p^{\mu_i p^l + \lambda_i l + \nu_i} = p^{\lambda_i l + \nu_i}$$

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- *Fix an odd* $i > 1$. *Under some conditions,*

$$\lambda_i \geq 2 \Longleftrightarrow \chi \cup \alpha_i = 0$$

Where α_i *is an element* K^* *constructed from* $\varepsilon_i \mathrm{Cl}(K)[p]$

Gold's criterion

Theorem (Gold's criterion[Gol74])

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- *Then*

$$\lambda \geq 2 \iff \alpha^{p-1} \equiv 1 \mod \tilde{\mathfrak{P}}_0^2$$

Here α *is a generator of* $\mathfrak{P}_0^{h_K}$

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By some work, easy to see:

$$\alpha^{p-1} \equiv 1 \mod \tilde{\mathfrak{P}}_0^2 \Longleftrightarrow log_p \alpha \equiv 0 \mod p^2$$

Here log_p is the p-adic log.

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If we work harder, by Poitou-Tate duality

$$log_p \alpha \equiv 0 \mod p^2 \iff \chi \cup \alpha = 0$$

Comparing

Theorem (McCallum-Sharifi[MS03])

Let K be a cyclotomic field $\mathbb{Q}(\mu_p)$. For cyclotomic $\mathbb{Z}_p$ extensions, under some conditions:

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for odd $i > 1$.

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Remark

Both theorems has the form "$\lambda \geq 2 \Longleftrightarrow \chi \cup \alpha = 0$", which motivates us to find the deep reason behind it.

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Massey products

Slogan

Massey product is a generalization of cup products.

- Given $\chi_1, \chi_2 \in H^1(G, \mathbb{F}_p) \cong \mathrm{Hom}(G, \mathbb{F}_p)$, we can form two representations $G \to GL_2(\mathbb{F}_p)$:

$$\rho_{\chi_1}(g) = \begin{pmatrix} 1 & \chi_1(g) \\ 0 & 1 \end{pmatrix}, \rho_{\chi_2}(g) = \begin{pmatrix} 1 & \chi_2 g \\ 0 & 1 \end{pmatrix}$$

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- Generally if we have a bunch of representations derived from elements in $H^1(G, \mathbb{F}_p)$ and they are compatible in a certain way, Massey products are the obstruction for us to glue them.

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- $\chi_1 \cup \phi_{2,3} + \phi_{1,2} \cup \chi_3 \in H^2(G, \mathbb{F}_p)$ is the Massey products of (χ_1, χ_2, chi_3) with respect to the defining system M.

proper defining system

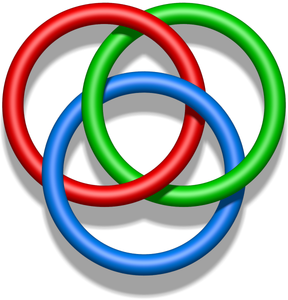
A defining system is called proper defining system if it is of the following form:

$$\begin{bmatrix} 1 & \chi & \binom{\chi}{2} & \binom{\chi}{3} & \binom{\chi}{4} & \cdots & * \\ 0 & 1 & \chi & \binom{\chi}{2} & \binom{\chi}{3} & \cdots & \psi_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & \chi & \binom{\chi}{2} & \psi_2 \\ 0 & 0 & 0 & 0 & 1 & \chi & \psi_1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \psi_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Here $\binom{n}{d} = \frac{n!}{d!(n-d)!}$.

Massey products and knots

There is an analogy between knots and primes in which $H^*(G_{K,S}, \mathbb{F}_p)$ plays a similar role as the cohomology of knot complements. Massey products were first introduced by Massey when considering the following knots. Cup products (i.e. linking numbers in knot theory) of any two rings are all zero. Hence cup products fail to determine whether the following knots are trivial. However, the triple Massey product of three rings is not zero, which tells us three rings are linked in a nontrivial way.



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Generalized Bockstein Map

- $G_{K,S}/G_{K_\infty,S} \cong \mathrm{Gal}(K_\infty/K) \cong \mathbb{Z}_p$
- Let σ be a topological generator of $G_{K,S}/G_{K_\infty,S}$.
- Define the complete algebra $\Omega := \mathbb{F}_p[[G_{K,S}/G_{K_\infty,S}]]$
- Let $I =< \sigma - 1 >$ be the augmentation ideal.
- we have an exact sequence :

$$0 \to \mathbb{F}_p \cong I^n/I^{n+1} \to \Omega/I^{n+1} \to \Omega/I^n \to 0$$

- After tensor with μ_p, it is still exact.

$$0 \to \mu_p \cong \mu_p \otimes I^n/I^{n+1} \to \mu_p \otimes \Omega/I^{n+1} \to \mu_p \otimes \Omega/I^n \to 0$$

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- The connecting map $\Psi^{(n)} : H^1(G_{K,S}, \mu_p \otimes \Omega/I^n) \to H^2(G_{K,S}, \mu_p \otimes I^n/I^{n+1}) = H^2(G_{K,S}, \mu_p)$ is called the generalized Bockstein map.

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Main result

Theorem (Q.)

- *Let* $K \subset K_1 \subset K_2 \subset \cdots \subset K_\infty$ *be a* $\mathbb{Z}_p$ *extension of* K
- *Let* S *be the set of primes above* p *for* K
- K_∞/K *is totally ramified for all primes in* S.
- *Let* $X_{cs} = \varprojlim \mathrm{Cl}_S(K_l)$ *and* μ_{cs}, λ_{cs} *be the Iwasawa invariant of* X_{cs}.
- *Assume* X_{cs} *has no torsion element and* $H^2(G_{K,S}, \mu_p) \cong \mathbb{F}_p$.

Then $\mu_{cs} = 0$ *if and only if there exists* k *such that* $\Psi^{(k)} \neq 0$ *for some* k.
If $\mu_{cs} = 0$, *then* $\lambda_{cs} = min\{n|\Psi^{(n)} \neq 0\} - \#S + 1$

Applying to the same case as Gold's criterion

Corollary (Q.)

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- *Assume $\lambda \geq n-1$*
- *Then $\lambda \geq n \Leftrightarrow n$-fold Massey product $(\chi, \chi, \cdots \chi, \alpha) = 0$ with respect to a proper defining system. Here α is a generator of $\mathfrak{P}_0^{h_K}$*

Applying to the same case as McCallum-Sharifi's result

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- *Fix an odd* $i > 1$. *Under some conditions, assume* $\lambda_i \geq n - 1$. *Then* $\lambda_i \geq n \Leftrightarrow n$*-fold Massey product* $\varepsilon_i(\chi, \chi, \cdots \chi, \alpha_i) = 0$ *with respect to a proper defining system, where* α_i *is an element* K^* *constructed from* $\varepsilon_i \mathrm{Cl}(K)[p]$

Idea of proof

- Use Kummer theory to connect the size of class groups and cohomological groups.

$$0 \to \mathcal{O}_{K,S}^*/(\mathcal{O}_{K,S}^*)^p \to H^1(G_{K,S}, \mu_p) \to \mathrm{Cl}_S(K)[p] \to 0$$

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- The size of cohomological groups is controlled by generalized Bockstein map [LLS+23].

$$\frac{I^n H^2_{\mathrm{Iw}}(G_{K_\infty,S}, \mu_p)}{I^{n+1} H^2_{\mathrm{Iw}}(G_{K_\infty,S}, \mu_p)} \cong \frac{H^2(G_{K,S}, \mu_p) \otimes I^n/I^{n+1}}{\mathrm{Im}\,\Psi^{(n)}}$$

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- Under some conditions, the image of generalized Bockstein map is spanned by Massey products[LLS+23].

THANK YOU!

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